

# The Orbit Method for Finite Groups of Nilpotency Class Two of Odd Order

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February 1, 2008

## Abstract

First, I construct an isomorphism between the categories of (topological) groups of nilpotency class 2 with 2-divisible center and (topological) Lie rings of nilpotency class 2 with 2-divisible center. That isomorphism allows us to construct adjoint and coadjoint representations as usual. For a finite group  $G$  of nilpotency class 2 of odd order, I construct a basis in its group algebra  $\mathbb{C}[G]$ , parameterized by elements of  $\mathfrak{g}^*$  so that the elements of coadjoint orbits form bases of simple two-side ideals of  $\mathbb{C}[G]$ . That construction gives us a one-to-one correspondence between  $G$ -orbits in  $\mathfrak{g}^*$  and classes of equivalence of irreducible unitary representations of  $G$ , implying a very simple character formula. The properties of that correspondence are similar to the properties of the analogous correspondence given by Kirillov's orbit method for nilpotent connected and simply connected Lie groups. The diagram method introduced in my article [1] and my thesis [2], gives us a convenient way to study normal forms on the orbits and corresponding representations.

## 1 Introduction

Fortunately, Kirillov published recently a survey of merits and demerits of the orbit method [3], so I can refer to it and skip a historical introduction here.

## 2 Groups of nilpotency class 2

By definition, a group  $B$  of nilpotency class 2 is a central extension of an abelian group, i.e. there is an exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \quad (2.1)$$

where  $A$  and  $C$  are abelian groups and  $A$  is the center of  $B$ . In other words, elements of the group  $B$  of nilpotency class 2 can be written as pairs  $b = (a, c)$  with  $a \in A$  and  $c \in C$  so that

$$(a_1, c_1) \cdot (a_2, c_2) = (a_1 + a_2 + \psi(c_1, c_2), c_1 + c_2). \quad (2.2)$$

The associativity of group operation (2.2) is equivalent to the following identity:

$$\psi(c_1, c_2) + \psi(c_1 + c_2, c_3) = \psi(c_1, c_2 + c_3) + \psi(c_2, c_3) \quad (2.3)$$

which means that  $\psi$  is a 2-cocycle,  $\psi \in C^2(C, A)$  supposing that the action of  $C$  on  $A$  is trivial. Note that substituting either  $c_1 = c_2 = 0$ , or  $c_2 = c_3 = 0$  in (2.3), we obtain the identities

$$\psi(0, c) = \psi(c, 0) = \psi(0, 0). \quad (2.4)$$

For every such cocycle  $\psi$ , the element  $(-\psi(0, 0), 0)$  is an identity of  $B$  and

$$(a, c)^{-1} = (-a - \psi(c, -c) - \psi(0, 0), -c) \quad (2.5)$$

is a left inverse to  $(a, c)$ , which means that formula (2.2) defines a group structure in set  $B = A \times C$  for every cocycle  $\psi \in C^2(C, A)$ . In particular, the left inverse (2.5) is a right inverse of  $(a, c)$  as well, thus formula (2.3) implies the identity

$$\psi(c, -c) = \psi(-c, c). \quad (2.6)$$

In this construction, both mappings

$$A \rightarrow A \times \{0\}, \quad a \mapsto (a - \psi(0, 0), 0); \quad (2.7)$$

$$C \rightarrow B/A, \quad c \mapsto (0, c)A \quad (2.8)$$

are isomorphisms. Choosing other representatives of cosets than  $(0, c)$  in (2.8), for any function  $q : C \rightarrow A$ ,

$$C \rightarrow B/A, \quad c \mapsto (0, c)(q(c) - \psi(0, 0), 0)A = (q(c), c)A \quad (2.9)$$

we obtain the same group structure in  $B$ . Renaming

$$(a, c)_{\text{new}} = (a + q(c), c), \quad (2.10)$$

we obtain from (2.2) that the same group structure in  $B$  can be defined by 2-cocycle

$$\psi_{\text{new}}(c_1, c_2) = \psi(c_1, c_2) + q(c_1) + q(c_2) - q(c_1 + c_2). \quad (2.11)$$

The difference between these old and new cocycles is equal to the coboundary of 1-chain  $q$ , thus the group structure in  $B$  is uniquely determined by 2-cocycles modulo coboundaries of 1-chains, i.e. by elements of  $H^2(C, A)$  with trivial action of  $C$  on  $A$ .

**Definition 2.1.** We'll call a cocycle  $\psi \in C^2(C, A)$  *centered* iff  $\psi(0, 0) = 0$ .

From (2.4), for a centered cocycle  $\psi$ , for every  $c \in C$ ,

$$\psi(0, c) = \psi(c, 0) = \psi(0, 0) = 0. \quad (2.12)$$

Choosing  $q(0) = -\psi(0, 0)$  in (2.11), we obtain a centered cocycle. Comparing (2.12) with the formula for the identity in  $B$ , we see that we can always choose a centered cocycle  $\psi$  in the same cohomology class so the identity element of  $B$  was  $(0, 0)$ .

**Definition 2.2.** An abelian group  $A$  is *2-divisible* if homomorphism

$$A \rightarrow A, \quad a \mapsto a + a \quad (2.13)$$

is an automorphism. For an element  $a$  of a 2-divisible abelian group, we denote  $a/2$  the image of  $a$  under the automorphism inverse to (2.13).

**Lemma 2.3.** *A finite abelian group is 2-divisible iff it has an odd order. An abelian  $p$ -group is 2-divisible iff  $p$  is odd.*

*Proof.* The kernel of (2.13) is the subgroup of elements of order 2. So abelian 2-groups and abelian finite groups of even order can't be 2-divisible. Finite abelian groups of odd order don't have elements of order 2, so homomorphism (2.13) is injective, thus its image contains as many elements as  $A$  does, so it is  $A$ . For abelian  $p$ -groups with odd  $p$ , homomorphism (2.13) is injective, and its restriction on every cyclic subgroup is an automorphism of that subgroup, so (2.13) is surjective as well.  $\square$

**Definition 2.4.** We'll call a cocycle  $\psi \in C^2(C, A)$  *equalized* iff  $\psi(c, -c) = 0$  for all  $c \in C$ .

For an equalized cocycle  $\psi$ , from (2.5), we have

$$(a, c)^{-1} = (-a, -c) \quad (2.14)$$

for every element  $(a, c) \in B$ .

**Lemma 2.5.** *For a 2-divisible abelian group  $A$  and an abelian group  $C$ , every element of  $H^2(C, A)$  can be represented by an equalized cocycle.*

*Proof.* If  $A$  is a 2-divisible abelian group, for a centered cocycle  $\psi$ , choosing  $q(c) = -\psi(c, -c)/2$  in (2.11), and combining it with (2.6), we obtain an equalized cocycle.  $\square$

For any  $A$ , if  $C$  doesn't have elements of order 2, an equalized cocycle can be constructed from a centered cocycle  $\psi$  by choosing  $\{q(c), q(-c)\} = \{-\psi(c, -c), 0\}$  with an arbitrary choice of which of them must be equal 0. Here is the example showing that it can't be obtained for all cases.

**Example 2.6.** Let either  $B \simeq D_8$ , a dihedral group of order 8, or  $B \simeq Q_8$ , a quaternionic group of order 8. In both cases  $A \simeq C_2$  and  $C \simeq C_2 \oplus C_2$  where  $C_n$  denotes a cyclic group of order  $n$ . We have  $a = -a$  and  $c = -c$  for all elements  $a \in A$ ,  $c \in C$ . If formula (2.14) was true, then all nontrivial elements of  $B$  would have order 2, which is false.

If we have a centered or an equalized cocycle and we want to change it adding the coboundary of a 1-chain  $q$  so that the new cocycle was still centered or equalized, we need the following conditions on  $q$ :  $q(0) = 0$  for centered cocycles, or  $q(-c) = -q(c)$  for all  $c \in C$  for equalized cocycles. Note also that for an equalized cocycle  $\psi$  for any  $c_1, c_2 \in C$ ,

$$-\psi(c_1, c_2) = \psi(-c_2, -c_1), \quad (2.15)$$

that follows from the identity  $(b_1 \cdot b_2)^{-1} = b_2^{-1} \cdot b_1^{-1}$ .

All what was told from (2.1) until now, was applicable to all central group extensions. We didn't explore the fact that  $A$  is exactly the center of  $B$ . What do we need for that? We need that cocycle  $\psi$  was non-degenerate.

**Definition 2.7.** A cocycle  $\psi$  is *non-degenerate* iff for every  $c_1 \in C$ ,  $c_1 \neq 0$  there exist such  $c_2 \in C$  that

$$\psi(c_1, c_2) \neq \psi(c_2, c_1). \quad (2.16)$$

For a non-degenerate cocycle  $\psi$ , element  $b = (a_1, c_1) \in B$  with  $c_1 \neq 0$  can't be central since it is not commute with  $(0, c_2)$  with  $c_2$  satisfying (2.16). Every coboundary of a 1-chain  $q$  is symmetric, so adding it to  $\psi$  doesn't change inequality (2.16). That means that all cocycles representing a cohomology class from  $H^2(C, A)$ , are either non-degenerate, in which case we'll call that class non-degenerate; or degenerate, then we'll call that class degenerate as well.

### 3 Lie rings of nilpotency class 2

By a Lie ring I mean an abelian group with a bilinear commutator  $[., .]$  such that  $[a, a] = 0$  for all  $a$ , it implies  $[a, b] = -[b, a]$ , and satisfying Jacobi identity. A commutative Lie ring means zero commutator.

By definition, a Lie ring  $\mathfrak{b}$  of nilpotency class 2 is a central extension of a commutative Lie ring, i.e. there is an exact sequence

$$0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{b} \rightarrow \mathfrak{c} \rightarrow 0 \quad (3.1)$$

where  $\mathfrak{a}$  and  $\mathfrak{c}$  are commutative Lie rings and  $\mathfrak{a}$  is the center of  $\mathfrak{b}$ . In other words, elements of the Lie ring  $\mathfrak{b}$  of nilpotency class 2 can be written as pairs  $b = (a, c)$  with  $a \in \mathfrak{a}$  and  $c \in \mathfrak{c}$  so that

$$(a_1, c_1) + (a_2, c_2) = (a_1 + a_2 + \phi(c_1, c_2), c_1 + c_2). \quad (3.2)$$

and

$$[(a_1, c_1), (a_2, c_2)] = [\eta(c_1, c_2) - \phi(0, 0), 0]. \quad (3.3)$$

The associativity and commutativity of the operation in  $\mathfrak{b}$  defined in (3.2) are equivalent to  $\phi$  being a symmetric 2-cocycle  $\phi \in C^2(C, A)$  where  $C$  and  $A$  are underlying abelian groups of  $\mathfrak{c}$  and  $\mathfrak{a}$ , correspondingly, supposing that the action of  $C$  on  $A$  is trivial. The commutator properties are equivalent to the fact that  $\eta : C \times C \rightarrow A$  is a skew-symmetric bihomomorphism where skew-symmetric means  $\eta(c, c) = 0$  for all  $c \in C$  which implies  $\eta(c_1, c_2) = -\eta(c_2, c_1)$ . In other words,  $\eta$  is a 2-cocycle  $\eta \in C^2(\mathfrak{c}, A)$  with trivial action of  $\mathfrak{c}$  on  $A$ .

In this construction, both mappings

$$\mathfrak{a} \rightarrow \mathfrak{a} \times \{0\}, \quad a \mapsto (a - \phi(0,0), 0); \quad (3.4)$$

$$\mathfrak{c} \rightarrow \mathfrak{b}/\mathfrak{a}, \quad c \mapsto (0, c) + \mathfrak{a} \quad (3.5)$$

are Lie ring isomorphisms. Choosing other representatives of cosets than  $(0, c)$  in (3.5), for any function  $q : C \rightarrow A$ ,

$$\mathfrak{c} \rightarrow \mathfrak{b}/\mathfrak{a}, \quad c \mapsto (0, c) + (q(c) - \phi(0,0), 0) + \mathfrak{a} = (q(c), c) + \mathfrak{a} \quad (3.6)$$

we obtain the same Lie ring structure in  $\mathfrak{b}$ . Renaming

$$(a, c)_{\text{new}} = (a + q(c), c), \quad (3.7)$$

we obtain from (3.2) that the same Lie ring structure in  $\mathfrak{b}$  can be defined by 2-cocycles

$$\phi_{\text{new}}(c_1, c_2) = \phi(c_1, c_2) + q(c_1) + q(c_2) - q(c_1 + c_2), \quad (3.8)$$

obtaining from  $\phi$  by adding a 1-chain, and  $\eta$ . Note that the coboundary of a 1-chain  $r \in C^1(\mathfrak{c}, A)$  with the trivial action of  $\mathfrak{c}$  on  $A$ , is  $r(c_1, c_2) = r([c_1, c_2]) = 0$ . Thus the Lie ring structure in  $\mathfrak{b}$  is uniquely determined by elements of  $H_{\text{sym}}^2(C, A) \oplus H^2(\mathfrak{c}, A)$  with trivial actions of  $C$  and  $\mathfrak{c}$  on  $A$ , where  $H_{\text{sym}}^2(C, A)$  denotes the subgroup of 2-cohomology classes defined by symmetric cocycles. The same as in the previous section, choosing  $q(c) = -\phi(0,0)$  in (3.8), we get  $\phi_{\text{new}}(0,0) = 0$ , and  $(0,0) = 0 \in \mathfrak{b}$ .

All what was told from (3.1) until now, was applicable to all central Lie ring extensions. We didn't explore the fact that  $\mathfrak{a}$  is exactly the center of  $\mathfrak{b}$ . What do we need for that? We need that cocycle  $\eta$  was non-degenerate.

**Definition 3.1.** A skew-symmetric bihomomorphism  $\eta : C \times C \rightarrow A$  is *non-degenerate* iff for every  $c_1 \in C$ ,  $c_1 \neq 0$  there exist such  $c_2 \in C$  that

$$\eta(c_1, c_2) \neq 0. \quad (3.9)$$

For a non-degenerate cocycle  $\eta$ , element  $b = (a_1, c_1) \in \mathfrak{b}$  with  $c_1 \neq 0$  can't be central since its commutator with  $(0, c_2)$  with  $c_2$  satisfying (3.9), is not 0.

## 4 Lie correspondence

**Theorem 4.1.** *For a 2-divisible (topological) abelian group  $A$  and an abelian (topological) group  $C$  acting trivially on  $A$ , the following mapping:*

$$L_c : C^2(C, A) \rightarrow C_{\text{sym}}^2(C, A) \oplus C^2(\mathfrak{c}, A), \quad \psi \mapsto (\phi, \eta) \quad (4.1)$$

where  $\mathfrak{c}$  is the commutative (topological) Lie ring, underlying abelian group of which is  $C$ , acting trivially on  $A$ ,  $C_{\text{sym}}$  denotes symmetric cocycles, and

$$\phi(c_1, c_2) = \frac{\psi(c_1, c_2) + \psi(c_2, c_1)}{2}, \quad (4.2)$$

$$\eta(c_1, c_2) = \psi(c_1, c_2) - \psi(c_2, c_1), \quad (4.3)$$

is an isomorphism, factor of which by coboundaries of 1-chains is an isomorphism of the cohomology groups

$$L_h : H^2(C, A) \rightarrow H_{\text{sym}}^2(C, A) \oplus H^2(\mathfrak{c}, A). \quad (4.4)$$

For a 2-divisible (topological) commutative Lie ring  $\mathfrak{a}$  underlying abelian group of which is  $A$ , and a commutative (topological) Lie ring  $\mathfrak{c}$  acting trivially on  $A$ , the following mapping:

$$E_c : C_{\text{sym}}^2(C, A) \oplus C^2(\mathfrak{c}, A) \rightarrow C^2(C, A), \quad (\phi, \eta) \mapsto \psi \quad (4.5)$$

where

$$\psi(c_1, c_2) = \phi(c_1, c_2) + \frac{\eta(c_1, c_2)}{2}, \quad (4.6)$$

is an isomorphism, factor of which by coboundaries of 1-chains is an isomorphism of the cohomology groups

$$E_h : H_{\text{sym}}^2(C, A) \oplus H^2(\mathfrak{c}, A) \rightarrow H^2(C, A). \quad (4.7)$$

The isomorphisms  $L_c$  and  $E_c$  are mutually inverse and the isomorphisms  $L_h$  and  $E_h$  are mutually inverse. Cocycle  $\phi$  is centered or equalized iff  $\psi$  is centered or equalized, correspondingly. Cocycle  $\eta$  is non-degenerate iff  $\psi$  is non-degenerate.

*Proof.* From (4.2),  $\phi$  is a cocycle, by linearity, and it is symmetric,

$$\phi(c_1, c_2) = \phi(c_2, c_1). \quad (4.8)$$

$\eta$  is skew-symmetric by (4.3). To check that it is a bihomomorphism, add the following cocycle identities:

$$\psi(c_1 + c_2, c_3) + \psi(c_1, c_2) = \psi(c_1, c_2 + c_3) + \psi(c_2, c_3), \quad (4.9)$$

$$\psi(c_3 + c_1, c_2) + \psi(c_3, c_1) = \psi(c_3, c_1 + c_2) + \psi(c_1, c_2), \quad (4.10)$$

$$-\psi(c_1 + c_3, c_2) - \psi(c_1, c_3) = -\psi(c_1, c_3 + c_2) - \psi(c_3, c_2). \quad (4.11)$$

After cancelling equal items, we get

$$\psi(c_1 + c_2, c_3) + \psi(c_3, c_1) - \psi(c_1, c_3) = \psi(c_3, c_1 + c_2) + \psi(c_2, c_3) - \psi(c_3, c_2), \quad (4.12)$$

or

$$\eta(c_1 + c_2, c_3) = \eta(c_1, c_3) + \eta(c_2, c_3). \quad (4.13)$$

$L_c$  is a homomorphism by linearity of (4.2) and (4.3). Adding coboundary to  $\psi$  adds the same coboundary to  $\phi$ , and inverse from (4.6), and we don't have to worry about adding coboundaries to  $\eta$  since  $H^2(\mathfrak{c}, A) = C^2(\mathfrak{c}, A)$ , so  $L_c$  defines  $L_h$ . From the other side,  $\psi$  defined in (4.6) is a cocycle by linearity (note, that  $\eta$  is a cocycle since every bihomomorphism from  $C \times C$  to  $A$  is an element of  $C^2(C, A)$ ).  $E_c$  is a homomorphism by linearity of (4.6). It is easy to check that  $L_c \circ E_c = \text{id}$  and  $E_c \circ L_c = \text{id}$ . Since  $L_h$  and  $E_h$  are defined from  $L_c$  and  $E_c$  by a factorization by the same coboundaries, they are mutually inverse isomorphisms as well. The last statement of the theorem immediately follows from the definitions.  $\square$

For a group  $B$  of nilpotency class 2 with 2-divisible center denote  $L(B)$  the Lie ring of nilpotency class 2 defined on the underlying set of  $B$  by cocycles (4.2) and (4.3). Theorem 4.1 tells that this construction doesn't depend on the choice of the cocycle  $\psi$  defining a group structure in  $B$ . For a Lie ring  $\mathfrak{b}$  of nilpotency class 2 with 2-divisible center denote  $E(\mathfrak{b})$  the group of nilpotency class 2 defined on the underlying set of  $\mathfrak{b}$  by a cocycle (4.6). Theorem 4.1 tells that this construction doesn't depend on the choice of the cocycles  $\phi$  and  $\eta$  defining a Lie ring structure in  $\mathfrak{b}$ .

For a homomorphism  $f : B_1 \rightarrow B_2$  of groups  $B_1$  and  $B_2$  of nilpotency class 2 with 2-divisible centers denote  $L(f) = f$  considered as a function from  $L(B_1)$  to  $L(B_2)$ . Analogously, for a homomorphism  $f : \mathfrak{b}_1 \rightarrow \mathfrak{b}_2$  denote  $E(f) = f$  considered as a function from  $E(\mathfrak{b}_1)$  to  $E(\mathfrak{b}_2)$ .

**Theorem 4.2.**  *$L$  and  $E$  are mutually inverse functors defining an isomorphism between categories*

of groups of nilpotency class 2 with 2-divisible center and Lie rings of nilpotency class 2 with 2-divisible center.

*Proof.* By Theorem 4.1,  $L$  and  $E$  are mutually inverse if they are functors. All categorical properties would follow immediately from definitions if we showed that  $L(f)$  is a Lie ring homomorphism for every group homomorphism  $f$  and  $E(f)$  is a group homomorphism for every Lie ring homomorphism  $f$ .

By Lemma 2.5, we can choose equalized cocycles  $\psi_1$  and  $\psi_2$  defining group structures in  $B_1$  and  $B_2$ . Then symmetric cocycles  $\phi_1$  and  $\phi_2$  defined by Theorem 4.1, are equalized as well. Multiplying the left hand sides and the right hand sides of the following identities:

$$\begin{aligned} f(a_1 + a_2 + \psi_1(c_1, c_2), c_1 + c_2) &= f(a_1, c_1) \cdot f(a_2, c_2) \\ &= (\alpha_1, \gamma_1) \cdot (\alpha_2, \gamma_2) = (\alpha_1 + \alpha_2 + \psi_2(\gamma_1, \gamma_2), \gamma_1 + \gamma_2), \end{aligned} \quad (4.14)$$

$$\begin{aligned} f(-a_1 - a_2 - \psi_1(c_2, c_1), -c_1 - c_2) &= (f(a_2, c_2) \cdot f(a_1, c_1))^{-1} \\ &= ((\alpha_2, \gamma_2) \cdot (\alpha_1, \gamma_1))^{-1} = (-\alpha_1 - \alpha_2 - \psi_2(\gamma_2, \gamma_1), -\gamma_1 - \gamma_2) \end{aligned} \quad (4.15)$$

we get

$$f(\eta_1(c_1, c_2), 0) = (\eta_2(\gamma_1, \gamma_2), 0). \quad (4.16)$$

Dividing the central elements by 2 and inverting, we get

$$f\left(-\frac{\eta_1(c_1, c_2)}{2}, 0\right) = \left(-\frac{\eta_2(\gamma_1, \gamma_2)}{2}, 0\right). \quad (4.17)$$

Multiplying (4.17) and (4.14), we get

$$f(a_1 + a_2 + \phi_1(c_1, c_2), c_1 + c_2) = (\alpha_1 + \alpha_2 + \phi_2(\gamma_1, \gamma_2), \gamma_1 + \gamma_2). \quad (4.18)$$

So

$$L(f)(b_1 + b_2) = L(f)(b_1) + L(f)(b_2) \quad (4.19)$$

for  $b_1 = (a_1, c_1)$ ,  $b_2 = (a_2, c_2)$ . This formula together with (4.16) rewritten as

$$L(f)([b_1, b_2]) = [L(f)(b_1), L(f)(b_2)] \quad (4.20)$$

means that  $L(f)$  is a Lie ring homomorphism.

Similarly, for a Lie ring homomorphism  $f$ , multiplying (4.18) and the identity obtained from (4.17) by changing  $-$  to  $+$ , we get (4.14) which means that  $E(f)$  is a group homomorphism.  $\square$

**Lemma 4.3.** *For any elements  $b, b_1, b_2$  of a group  $B$  of nilpotency class 2 with 2-divisible center,*

$$b^{-1} = -b, \quad (4.21)$$

$$b_1 \cdot b_2 = b_1 + b_2 + \frac{[b_1, b_2]}{2}, \quad (4.22)$$

$$b_1 \cdot b_2 \cdot b_1^{-1} = b_2 + [b_1, b_2], \quad (4.23)$$

$$b_1 \cdot b_2 \cdot b_1^{-1} \cdot b_2^{-1} = [b_1, b_2]. \quad (4.24)$$

If  $b_1$  and  $b_2$  commute, then

$$b_1 \cdot b_2 = b_1 + b_2. \quad (4.25)$$

*Proof.* Formula (4.21) is true because by Lemma 2.5 we can choose an equalized cocycle  $\psi$  defining group structure in  $B$  and it corresponds by Theorem 4.1 to the equalized cocycle  $\phi$  defining additive group structure in  $L(B)$ . Formula (4.25) is true because  $\psi(c_1, c_2) = \phi(c_1, c_2)$  for commuting  $b_1 = (a_1, c_1)$  and  $b_2 = (a_2, c_2)$ . We already used (4.22) at the end of the proof of Theorem 4.2. Formula (4.24) telling that the group commutator in  $B$  coincides with the Lie ring commutator in  $L(B)$  is true since the product of left hand sides of formulas (4.14) and (4.15) equals left hand side of (4.16). Formula (4.23) can be obtained by right multiplication of both sides of (4.24) by  $b_2$  and using (4.25).  $\square$

**Definition 4.4.** A group  $B$  is 2-rootable if mapping

$$B \rightarrow B, \quad b \mapsto b^2 \quad (4.26)$$

is a bijection.

**Corollary 4.5.** *A finite group of nilpotency class 2 is 2-rootable iff it has an odd order. A  $p$ -group of nilpotency class 2 is 2-rootable iff  $p$  is odd.*

*Proof.* Groups of even order and 2-groups have elements of order 2, so the identity covers more than once by (4.26), and these groups can't be 2-rootable. Finite groups of odd order and  $p$ -groups with

odd  $p$  have 2-divisible center, by Lemma 2.3, so  $b^2 = b + b$  by (4.25), and Lemma 2.3 applied to  $L(B)$ , completes the proof.  $\square$

## 5 The orbit method

**Lemma 5.1.** *For a group  $B$  of nilpotency class 2 with 2-divisible center, formula*

$$\text{Ad}(b)(l) = L(b \cdot l \cdot b^{-1}) \quad (5.1)$$

*defines a structure of left  $B$ -module in the underlying abelian group of  $L(B)$ .*

*Proof.* Conjugation by  $b \in B$  is an automorphism of  $B$ , so  $\text{Ad}(b)$  is a Lie ring automorphism by Theorem 4.2. The formula

$$\text{Ad}(b_1) \circ \text{Ad}(b_2) = \text{Ad}(b_1 \cdot b_2) \quad (5.2)$$

follows directly from the definition (5.1).  $\square$

As usual, we'll call  $\text{Ad}$  adjoint representation of  $B$ . Denote  $L(B)^*$  the group of (unitary) characters of the underlying abelian group of  $L(B)$ . Define coadjoint representation  $\text{Ad}^*$  as the dual to  $\text{Ad}$  left action of  $B$  in  $L(B)^*$ :

$$\text{Ad}^*(b)(\chi)(l) = \chi(\text{Ad}(b^{-1})(l)). \quad (5.3)$$

**Lemma 5.2.** *For a group  $B$  of nilpotency class 2 with 2-divisible center, for every  $b \in B$ ,  $l \in L(B)$ ,  $\chi \in L(B)^*$ ,*

$$\text{Ad}(b)(l) = l + [b, l], \quad (5.4)$$

$$\text{Ad}^*(b)(\chi)(l) = \chi(l - [b, l]). \quad (5.5)$$

*Proof.* Both formulas follow from (4.23) and definitions (5.1) and (5.3).  $\square$

Denote  $\mathcal{O}(B)$  the set of orbits of the coadjoint representation of  $B$ .

**Theorem 5.3 (Orbit method).** *For a finite group  $B$  of nilpotency class 2 of odd order, elements*

$$\left( X_\chi = \sum_{l \in L(B)} \chi(l) l \right)_{\chi \in L(B)^*} \quad (5.6)$$

*form an orthonormal basis in  $\mathbb{C}[B]$ . For every orbit  $\Omega$  of the coadjoint representation of  $B$ , the subspace*

$$V_\Omega = \text{span}(X_\chi)_{\chi \in \Omega} \quad (5.7)$$

*is a two-side ideal of a group algebra  $\mathbb{C}[B]$ , the restriction of the regular representation of  $B$  in  $\mathbb{C}[B]$  on  $V_\Omega$  is isotypic, and*

$$\mathbb{C}[B] = \bigoplus_{\Omega \in \mathcal{O}(B)} V_\Omega \quad (5.8)$$

*is the decomposition of the regular representation of  $B$  in  $\mathbb{C}[B]$  in the direct sum of isotypic components.*

*Proof.* Elements (5.6) form an orthonormal basis in  $\mathbb{C}[L(B)]$  and dot products in  $\mathbb{C}[L(B)]$  and  $\mathbb{C}[B]$  are the same.

$$\begin{aligned} b X_\chi &= \sum_{l \in L(B)} \chi(l) bl = \sum_{g \in B} \chi(b^{-1}g) g = \sum_{g \in B} \chi \left( -b + g + \frac{[-b, g]}{2} \right) g \\ &= \chi(-b) \sum_{g \in B} \text{Ad}^* \left( \frac{b}{2} \right) (\chi)(g) g = \chi(-b) X_{\text{Ad}^*(b/2)(\chi)} \in V_\Omega. \end{aligned} \quad (5.9)$$

Similarly,

$$\begin{aligned} X_\chi b &= \sum_{l \in L(B)} \chi(l) lb = \sum_{g \in B} \chi(gb^{-1}) g = \sum_{g \in B} \chi \left( g - b + \frac{[g, -b]}{2} \right) g \\ &= \chi(-b) \sum_{g \in B} \text{Ad}^* \left( \frac{-b}{2} \right) (\chi)(g) g = \chi(-b) X_{\text{Ad}^*(-b/2)(\chi)} \in V_\Omega. \end{aligned} \quad (5.10)$$

So  $V_\Omega$  is a two-side ideal of  $\mathbb{C}[B]$ . Because  $X_\chi$  are orthogonal for different  $\chi$ , these ideals  $V_\Omega$  are orthogonal for different  $\Omega$ . Since  $\mathbb{C}[B]$  is a semi-simple associative algebra and every component in the direct sum of the isotypic components

$$\mathbb{C}[B] = \bigoplus_{\tau \in \hat{B}} \text{Iso}(\tau) \quad (5.11)$$

is simple, every  $V_\Omega$  is either one of these components, or a direct sum of a few of them. But the number of coadjoint orbits coincides with the number of conjugate classes of  $B$ , by Duality Lemma 5.4, i.e. with the number of irreducible unitary representations of  $B$ , i.e. with the number of components in the sum (5.11). Thus every  $V_\Omega$  coincides with isotypic component  $\text{Iso}(\tau) \simeq n\tau$  for an irreducible unitary representation  $\tau$  where  $n = \dim \tau$ , and different orbits correspond to different irreducible representations.  $\square$

**Lemma 5.4 (Duality Lemma).** *Let a finite abelian group  $M$  be a left  $G$ -module for a finite group  $G$ . Denote  $M^*$  the group of unitary characters of  $M$  and define the structure of a dual left  $G$ -module in  $M^*$  by formula*

$$g\chi(m) = \chi(g^{-1}m). \quad (5.12)$$

*Then the number of  $G$ -orbits in  $M^*$  is the same as the number of  $G$ -orbit in  $M$ .*

*Proof.* Denote  $T_M$  the representation of  $G$  in  $\mathbb{C}[M]$  defined by formula  $T_M(g)m = gm$ . If element

$$x = \sum_{m \in M} x_m m \quad (5.13)$$

is an invariant of  $T_M$ , then for every orbit  $\Xi$  of  $G$  in  $M$  all coefficients  $x_m$  with  $m \in \Xi$  are the same. That means that elements

$$e_\Xi = \sum_{m \in \Xi} m \quad (5.14)$$

form a basis in the space of invariants of  $T_M$ , and the dimension of the space of invariants equals number of  $G$ -orbits in  $M$ . The same is true for  $M^*$ : the number of  $G$ -orbits in  $M^*$  equals the dimension of the space of invariants of representation  $T_{M^*}$  defined by formula  $T_{M^*}(g)\chi = g\chi$ . By construction, the space  $\mathbb{C}[M^*]$  is the dual space to  $\mathbb{C}[M]$  and representations  $T_M$  and  $T_{M^*}$  are dual. Thus if

$$T_M \simeq \bigoplus_{\tau \in \hat{G}} n_\tau \tau \quad (5.15)$$

is the decomposition of  $T_M$  in the sum of irreducible representations, then

$$T_{M^*} \simeq \bigoplus_{\tau \in \hat{G}} n_\tau \tau^* \quad (5.16)$$

is the decomposition of  $T_{M^*}$  in the sum of irreducible representations. The dimensions of the spaces of the invariants equal to multiplicities of the trivial representation in (5.15) and (5.16), which are the same since the trivial representation is self-dual.  $\square$

Formulas (5.8) and (5.11) set two different isomorphisms between the set of coadjoint orbits  $\mathcal{O}$  and the set of classes of equivalency of irreducible unitary representations  $\hat{B}$ , depending on what regular representation of  $B$  in  $C(B)$  we use.

**Definition 5.5.** Denote  $\tau(\Omega)$  and  $\Omega(\tau)$  the irreducible unitary representation class and coadjoint orbit so that

$$V_{\Omega(\tau)} = \text{Iso}(\tau(\Omega)) \simeq n\tau(\Omega) \quad (5.17)$$

where  $n = \dim \tau(\Omega)$  and the regular representation of  $B$  in  $\mathbb{C}[B]$  used in (5.11) and (5.17), is defined as

$$R(g)x = xg^{-1}. \quad (5.18)$$

We need to use the regular representation given by (5.18) to ensure for abelian  $B$  for an orbit containing one character, correspondence to that character.

**Corollary 5.6 (Dimension formula).** *For a finite group  $B$  of nilpotency class 2 of odd order, every coadjoint orbit  $\Omega$  of  $B$  has  $n^2$  elements where  $n = \dim \tau(\Omega)$ . In other words,*

$$\dim \tau(\Omega) = \sqrt{\#\Omega(\tau)}. \quad (5.19)$$

*Proof.* From (5.7),

$$\dim V_\Omega = \#\Omega. \quad (5.20)$$

Formula (5.19) follows directly from here and (5.17)  $\square$

**Lemma 5.7 (Stabilizer lemma).** *For a group  $B$  of nilpotency class 2 with 2-divisible center, for any coadjoint orbit  $\Omega$  and  $\chi_1, \chi_2 \in \Omega$*

$$\text{Stab } \chi_1 = \text{Stab } \chi_2 \quad (5.21)$$

and

$$\chi_1(b) = \chi_2(b) \quad (5.22)$$

for any  $b \in \text{Stab } \chi_1 = \text{Stab } \chi_2$  where  $\text{Stab } \chi$  denotes the stabilizer of  $\chi$ .

*Proof.* If  $b \in \text{Stab } \chi_1$ , then for all  $l \in L(B)$

$$\chi_1(l) = \chi_1(l - [b, l]) = \chi_1(l)/\chi_1([b, l]) \quad (5.23)$$

so  $\chi_1([b, l]) = 1$ . Since  $\chi_1$  and  $\chi_2$  are on the same orbit, there is such  $g \in B$  that for all  $l \in L(B)$

$$\chi_2(l) = \chi_1(l - [g, l]). \quad (5.24)$$

Then

$$\chi_2([b, l]) = \chi_1([b, l] - [g, [b, l]]) = \chi_1([b, l]) = 1 \quad (5.25)$$

so

$$\chi_2(l - [b, l]) = \chi_2(l)/\chi_2([b, l]) = \chi_2(l) \quad (5.26)$$

so  $b \in \text{Stab } \chi_2$  and we proved (5.21). Now

$$\chi_2(b) = \chi_1(b - [g, b]) = \chi_1(b)\chi_1([b, g]) = \chi_1(b) \quad (5.27)$$

□

**Theorem 5.8 (Character formula).** *For a finite group  $B$  of nilpotency class 2 of odd order, for any  $b \in B$  and  $\chi \in \Omega$*

$$\text{char } \tau(\Omega)(b) = \begin{cases} n\chi(b) & \text{if } b \in \text{Stab } \chi, \\ 0 & \text{otherwise,} \end{cases} \quad (5.28)$$

where  $n = \dim \tau(\Omega)$ .

*Proof.* Find the character of the restriction of regular representation (5.18) to  $V_\Omega$ . From (5.10), if  $b/2 \in \text{Stab } \chi$ , then from Stabilizer Lemma 5.7,  $b$  acts in  $V_\Omega$  by scalar multiplication on  $\chi(b)$ , so

$$\text{char } n\tau(\Omega)(b) = \dim V_\Omega \chi(b) = n^2 \chi(b) \quad (5.29)$$

Note that the cyclic subgroup generated by  $b$  is of odd order, so it contains  $b/2$ , that means that  $b$  and  $b/2$  either belong to  $\text{Stab } \chi$ , or not, simultaneously. Now, if  $b \notin \text{Stab } \chi$ , then again from (5.10) and from Stabilizer Lemma 5.13, all diagonal elements of the matrix of the action of  $b$  in  $V_\Omega$  are zeroes, so

$$\text{char } n\tau(\Omega)(b) = 0 \quad (5.30)$$

Dividing (5.29) and (5.30) by  $n$ , we get (5.28).  $\square$

**Corollary 5.9.** *For a finite abelian group  $B$  of odd order, for  $\Omega = \{\chi\}$ ,*

$$\tau(\Omega) = \chi \quad (5.31)$$

*Proof.* It follows from (5.30) for  $n = 1$  and  $\text{Stab } \chi = B$ .  $\square$

**Corollary 5.10.** *For a finite group  $B$  of nilpotency class 2 of odd order,*

$$\tau(-\Omega) = \tau(\Omega)^*. \quad (5.32)$$

*Proof.* Again it follows directly from (5.28).  $\square$

*Acknowledgments.* I would like to thank Martin Isaacs, Wolfgang Kappe and Peter Morris for useful discussions.

## References

- [1] ALEKSANDRS MIHAILOVS. Diagrams of Representations, math.RT/9803079 (1998).
- [2] ALEKSANDRS MIHAILOVS. *A Combinatorial Approach to Representations of Lie Groups and Algebras*. PhD thesis, University of Pennsylvania, 1998.
- [3] A. A. KIRILLOV. Merits and Demerits of the Orbit Method, *Bull. of the Amer. Math. Soc.* **36**, No. 4, p. 433–488 (1999).